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CHARACTERISTICS OF THE SOLITARY WAVE

By James W. Daily, Assoc. M. ASCE, and Samuel C. Stephan, Jr., Jun. M. ASCE

HYDRAULICS DIVISION

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PAPERS

CHARACTERISTICS OF THE SOLITARY WAVE

By James W. Daily,[1] Assoc. M. ASCE, and
Samuel C. Stephan, Jr.,[2] Jun. M. ASCE

Synopsis

The existence of the solitary wave was first reported by J. Scott Russell in 1838. This wave consists of a single elevation that seems to be propagated at constant velocity and unaltered in form. The solitary wave has assumed importance because of the occurrence under a variety of natural circumstances of solitary-like waves whose behavior apparently approximates that of the solitary wave. This similarity suggests the possibility of predicting the effects of these waves through knowledge of the characteristics of the solitary wave. For the purpose of such predictions, there is available a fund of theoretical treatments with some experimental confirmation, mostly based on very early investigations. There has been some renewed interest in checking experimentally the preciseness of the theory to describe certain of the characteristics, and a program of research has been undertaken by the Massachusetts Institute of Technology, in Cambridge, Mass. The purpose of this paper is to describe the solitary wave and the range of its occurrence and to summarize its theoretically determined characteristics and the status of their experimental verification.

Description of the Solitary Wave

Gravity waves can be broadly classified as either oscillatory or translatory. Oscillatory waves are periodic in character, imparting to the liquid an undulating motion with both horizontal and vertical components without causing appreciable permanent displacement. Translatory waves, on the contrary, cause a net displacement (translation) of the liquid in the direction of the wave motion. Oscillatory waves may occur in either relatively deep water affecting only the liquid near the surface, or in relatively shallow water affecting the

Note.—Written comments are invited for publication; the last discussion should be submitted by **June 1, 1952.**

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liquid over the full depth. A characteristic of translatory waves is that the fluid is set in motion over the full depth (with nearly constant velocity over any cross section at any instant). Thus translatory waves are, relatively speaking, shallow water waves. Oscillatory waves can be created by a periodic disturbance; translatory waves by a sudden permanent displacement (addition or subtraction) of a mass of liquid.

The solitary wave is a translation wave. Included in this classification are tidal waves, seismic waves, bore waves and other channel surges, and flood waves. Each of these several types of wave occurs under different circumstances and is propagated with different characteristics, depending on the relative importance of the gravity, friction, and inertia forces. For the flood wave, friction is predominant in controlling the motion. Tidal wave motion is essentially irrotational, with frictional damping producing only a secondary effect. Solitary wave motion also is essentially irrotational. Between these extremes, the several intermediate types form a continuous transition.

Translation waves of finite height or amplitude are in general propagated with a celerity that depends on height and with a wave form that changes with time. This continuous deformation of the wave form is independent of any simultaneous effect of friction in absorbing energy and changing the form. The solitary wave is an exception in that it is apparently a wave of permanent form and constant celerity so long as friction effects can be ignored. It consists of a single elevation of liquid above the level of the originally undisturbed surface, as shown in Fig. 1. Theoretically, the wave extends to infinity on either side of the crest, but practically its length is limited. A passing wave accelerates the liquid from zero to a maximum velocity as the crest passes and then allows a deceleration to zero again, causing a horizontal displacement by an amount proportional to the wave volume. The major portion of the wave energy is concentrated within a narrow zone straddling the crest.

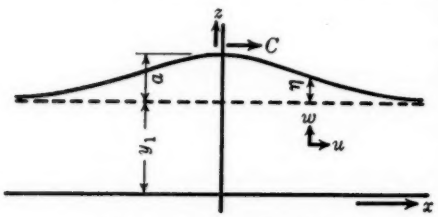

FIG. 1.—THE SOLITARY WAVE

Although the shape and other characteristics of oscillatory waves depend on wave length (the distance between two crests) as well as the wave height and the liquid depth,[3] the solitary wave characteristics are described completely by the wave height and liquid depth.

RANGE OF APPLICATION OF SOLITARY WAVE CHARACTERISTICS

J. Scott Russell[4,5] was responsible for calling attention to the solitary wave, in 1838, and for making the first investigations of its characteristics during the period from 1838 to 1845. He observed that acceleration or deceleration of

[3] "A Summary of the Theory of Oscillatory Waves," *Technical Report No. 2*, Beach Erosion Board, Corps of Engrs., U. S. Army, Washington, D. C., 1942.

[4] "Report of the Committee on Waves," by J. Scott Russell, Seventh Meeting of the British Association for the Advancement of Science, 1838, p. 417.

[5] "Report on Waves," by J. Scott Russell, Fourteenth Meeting of the British Association for the Advancement of Science, 1845, p. 311.

ships in navigation channels sometimes set up a wave with a solitary crest that then traversed the length of the channel, essentially unaltered in form and at constant velocity of propagation. He also observed that tidal waves behaved similarly in traversing the channels. The field observations were then supplemented with laboratory experiments that were undertaken primarily in consequence of an interest in waves as affecting navigation, although a broader significance was recognized at the time.

In the following fifty years, the solitary wave was the subject of considerable academic interest and theoretical study, with some additional experimental research. Then interest lagged until the growing concern over beach erosion and other shore-line problems, flood routing, and wave action in navigation inlets led to the application of the accumulated theoretical results to several practical problems.

Aside from the more obvious cases, such as tidal waves or artificially produced solitary crests traversing a channel, river, or canal, perhaps the most interesting application of knowledge of solitary wave characteristics is to the approximation of the behavior of periodic waves of long wave length in relatively shallow water. As oscillatory waves enter into shallow water, the crest increases in height and curvature, the trough becomes flatter, and the behavior of the individual crests becomes nearly independent of the wave length. This independence, as well as an observed physical similarity between the resulting crest and the solitary wave crest, led to the application of the characteristics of the solitary wave to the transformation and breaking of waves in shallow water, the rise of the sea surface accompanying high surf, the refraction of waves, the establishment of longshore currents, and the transport of sand in the surf zone. Walter H. Munk[6] has summarized the procedures and progress made along these lines.

Characteristics of the Solitary Wave

The key characteristics of the solitary wave are the wave celerity, wave form, and liquid particle velocities. With these known, the wave volume, wave energy, and amount of displacement of liquid can be determined as functions of depth and of the ratio of amplitude to depth. In addition the manner of energy loss through friction is an important consideration.

All theoretical solutions for the behavior of the solitary wave are based on the common conditions of irrotational two-dimensional motion of a liquid that satisfies the following boundary conditions: (1) The wave form is permanent; (2) the pressure over the free surface is constant; (3) the motion vanishes at infinity in both directions; and (4) the vertical motion vanishes at the channel bottom. Using the dynamical equations of motion, the velocity potential function can be expressed by an infinite series. Existing theories differ as to the type of series to be employed and the degree of approximation imposed by limiting the number of terms in the series when calculating each of the several characteristics. An exact solution has not been reported. This situation has resulted in certain differences in published formulas for celerity, profile shape, and other characteristics. These differences have been only partially reconciled by comparison with experimental evidence.

[6] "The Solitary Wave Theory and its Application to Surf Problems," by Walter H. Munk, *Annals of the New York Academy of Science*, Vol. 51, 1949, p. 376.

The most important results in this field are attributed to J. Boussinesq,[7] Lord Rayleigh,[8] and J. McCowan.[9] Rayleigh and Boussinesq employed a first approximation using only a few terms to establish expressions for the various characteristics. Mr. McCowan discussed solutions to a higher approximation with significant difference from the previous results in that the expressions for particle motion are more reasonable. In the excellent summary by Mr. Munk[6] the results of Boussinesq and Mr. McCowan were employed. From time to time, improvements in the theory have been suggested. Thus, A. Weinstein[10] proposed an expression for celerity carried to a higher approximation, and Joseph B. Keller[11] published a method of analysis in which the degree of approximation was more easily established and more easily extended, if desired. Garbis H. Keulegan and George Patterson[12] have reviewed the theory in detail and Mr. Keulegan[13] has discussed the effect of friction on damping.

This paper will summarize the most important of the theoretical principles and comment on their agreement with experimental results. For experimental confirmation, theoretical investigators have relied primarily upon the initial measurements of Mr. Russell and other measurements made a short time later by H. Bazin.[14] Generally this confirmation was restricted to the expression for celerity, the empirical form of which was similar to that obtained by Boussinesq and Rayleigh. The renewal of interest in the solitary wave has led to a reexamination of its characteristics in the laboratory in regard to agreement with theory and the effects of friction. Results of a program of research in the Hydrodynamics Laboratory at MIT will be used in this paper for comparison with the theory.

The laboratory program at MIT included measurements of celerity, wave form, particle motion, and the effects of channel surface friction in both horizontal and shoaling channels. However, data used for illustrations in the following sections include only measurements of celerity and profile shapes with a horizontal bottom. These experiments were made in a transparent walled channel 16.5 in. wide and 32 ft long. Waves were generated by a plunger that displaced a certain definite volume of liquid to cause the solitary elevated wave. The wave as initially generated is very close to the solitary form, and, after a travel of about 16 ft, it ceases to deform perceptibly, apparently reaching its final stable form. Celerity measurements are then made by timing the passage of the crest between two stations. The corresponding wave profile is obtained

[7] "Theorie Des Onde Et De Remous Qui Se Propagent Le Long D'un Canal Rectangulaire Horizontal, en Communiquant Au Liquide Contenn Dans Ce Canal Des Vitesses Sensiblement Parielles de La Surface Au Fond," by J. Boussinesq, *Journal of Mathematics*, Liouvilles, France, Vol. 17, 1872, p. 55.

[8] "On Waves," by Lord Rayleigh, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Vol. 1, 1876, p. 257.

[9] "On the Solitary Wave," by J. McCowan, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Vol. 32 (5), 1891, p. 45.

[10] "Sur La Vitesse De Propagation De L'onde Solitaire," by A. Weinstein, *Comptes Rendus*, Accademia del Linceі, Vol. 3, No. 8, 1926, p. 463.

[11] "The Solitary Wave and Periodic Waves in Shallow Water," by Joseph B. Keller, *Communications on Applied Mathematics*, Institute of Mathematics and Mechanics, New York University, New York, N. Y., Vol. I, No. 3, 1948, p. 286.

[12] "Mathematical Theory of Irrotational Translation Waves," by Garbis H. Keulegan and George W. Patterson, *Journal of Research*, National Bureau of Standards, Vol. 24, 1940, p. 47.

[13] "Gradual Damping of Solitary Waves," by G. H. Keulegan, *Journal of Research*, National Bureau of Standards, Vol. 40, 1948, p. 487.

[14] "Recherches Experimentales Sur La Propagation Des Ondes," by H. Bazin, Memoires Divers Savants a l'Academie des Sciences, Vol. 19, 1865, p. 495.

photographically. These preliminary measurements show considerable scatter that is caused by lack of precision in making these measurements rather than by any systematic error. Thus, despite this experimental scatter, certain trends are indicated that are felt to be reliable enough to justify discussion.

Celerity.—Mr. Russell reported that the velocity of wave propagation could be expressed by the relation

$$C = \sqrt{g(y_1 + a)} = \sqrt{g\,y_1}\left(1 + \frac{a}{y_1}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which C is the celerity of the wave, g is the acceleration of gravity, y_1 is the depth of the undisturbed water, and a is the wave amplitude (elevation of the crest above y_1). Later, Boussinesq, Rayleigh, and Mr. McCowan each obtained this equation theoretically as a "first approximation," as previously mentioned. In addition, Mr. McCowan pointed out that with slightly higher degrees of approximation, there could be obtained

$$C = \sqrt{g\,y_1}\left(1 + \frac{a}{y_1 + a}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$C = \sqrt{g\,y_1}\left(1 + \frac{a}{y_1 + \frac{19}{12}a}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Mr. Weinstein obtained, by a different method, the same equation carried to a higher degree of approximation

$$C = \sqrt{g\,y_1}\left[1 + \frac{a}{y_1} - \frac{21}{20}\left(\frac{a}{y_1}\right)^2\right]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

This equation is intended to apply for ratios of a/y_1 less than 0.2 whereas no such limit was specified for the previous equations. Eq. 1 indicated a linear relation between C^2 and the depth of liquid at the wave crest, while Eqs. 2, 3, and 4 each give increasingly large departures from the linear relation, as the ratio of a/y_1 increases.

The results of the early experiments of Mr. Bazin as well as those of Mr. Russell supported Eq. 1 to within an error of about 1%. The results of the measurements at MIT are shown in Fig. 2. In Fig. 2 observed celerities of a series of measurements are compared with the celerities as given by Eq. 1. This comparison indicates that the observed celerities are slightly less than the theoretical celerities. The total mass of this MIT data tends to substantiate this indication, as is shown by the center of gravity of the band of points in Fig. 2(*b*). None of the Eqs. 2, 3, or 4 appears to follow exactly the trend of experimental data although for values of a/y_1 less than 0.2 these equations each provide corrections in the right direction. For values of the term above 0.2 all the equations indicate increasingly large deviations from the measured values. For the full range of amplitude up to the point of breaking (important in surf problems), Eq. 1, possibly with a correction of about 0.5%, remains the best expression for the celerity.

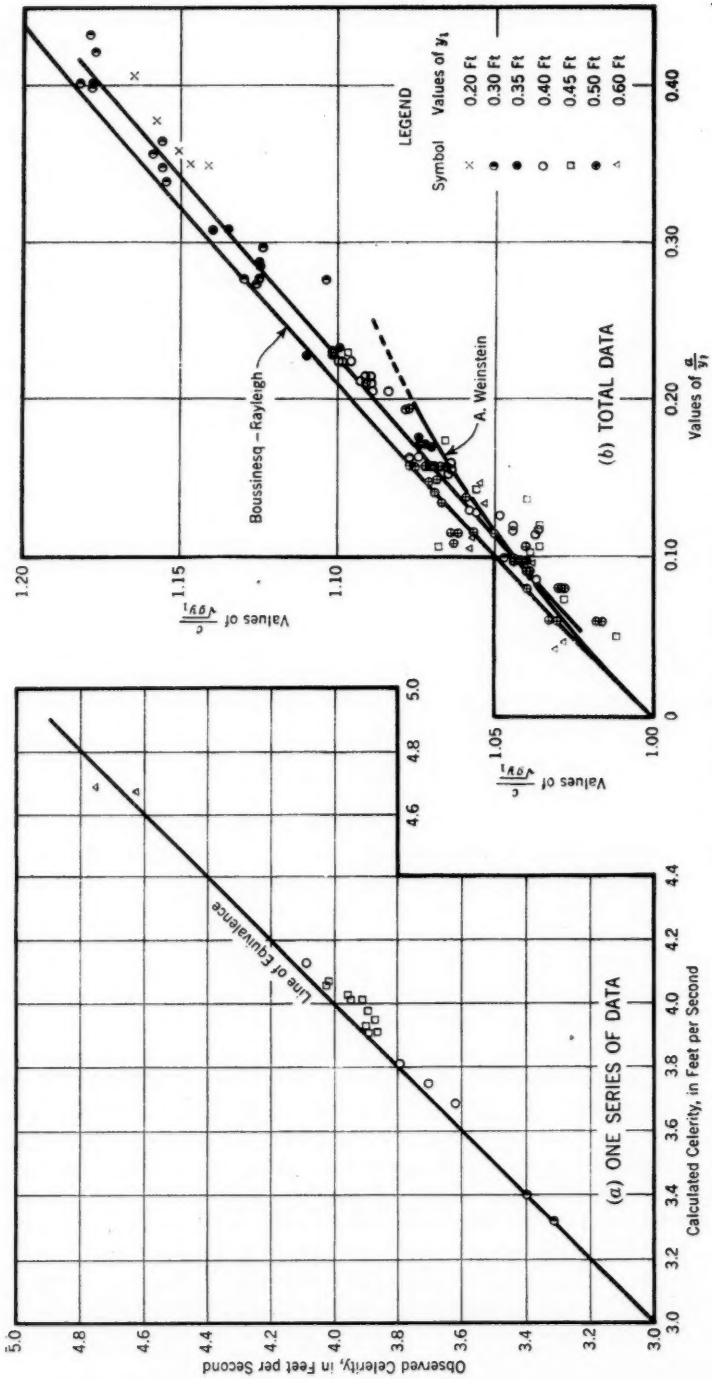

FIG. 2.—COMPARISON OF WAVE CELERITIES

Surface Profile.—The second report by Mr. Russell[5] contained traces of surface profiles that were described by a curve of versed sines. All theoretical solutions have indicated that the profile should be described by a hyperbolic secant curve. Mr. McCowan gave the solution as

$$\eta = a \operatorname{sech}^2 \left(\frac{m\,x}{2} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which η is the surface elevation above the undisturbed water level, a is amplitude as described, x is the horizontal distance measured from the crest

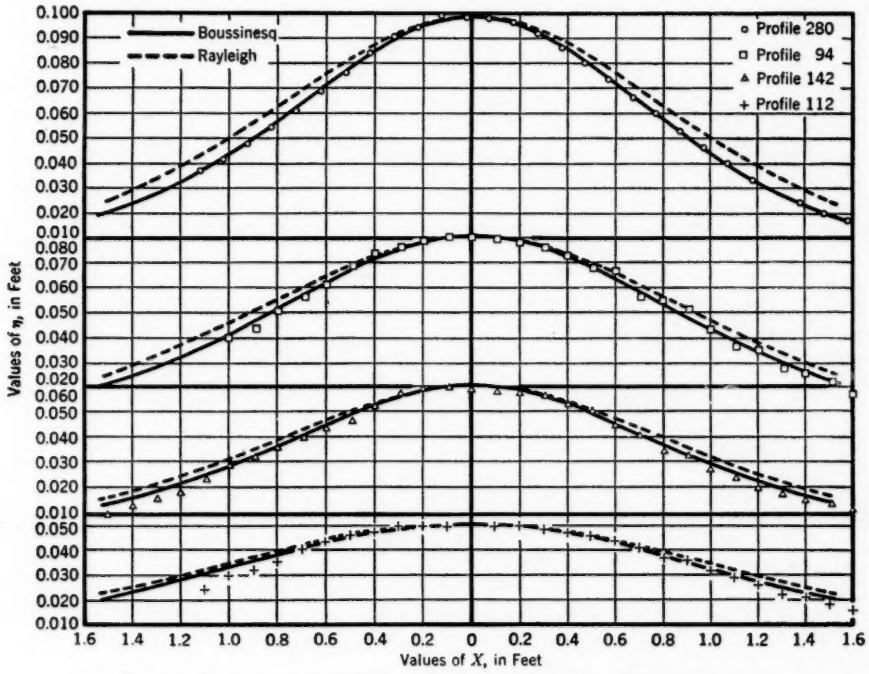

FIG. 3.—COMPARISON OF EXPERIMENTAL AND THEORETICAL WAVE PROFILES

position, and m is an expression that depends on the type of solution and degree of approximation. This value has been determined by various investigators, as follows:

Investigator	Value of m	
J. Boussinesq	$\sqrt{\frac{3\,a}{y_1^3}}$	(6*a*)
Lord Rayleigh	$\sqrt{\frac{3\,a}{y_1^2\,(y_1 + a)}}$	(6*b*)
J. McCowan	$\sqrt{\frac{3\,a}{y_1^2\left(y_1 + \frac{19}{12}\,a\right)}}$	(6*c*)

With each successive modification, the steepness of the wave front varies downward from the steepest form as given by Boussinesq. This is evidenced by the comparison between the plotted Boussinesq and Rayleigh equations in Fig. 3.

Actual waves should be expected to differ from these theoretical profiles in that the elevated surface will extend for only a finite distance from the crest. Practically, the degree of agreement near the crest is more important since the wave volume and energy are primarily concentrated there. Fig. 4 gives an indi-

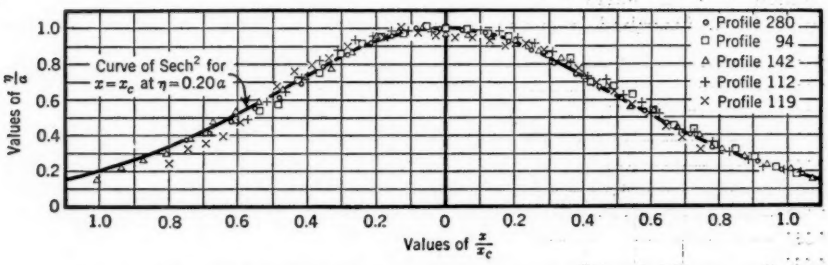

FIG. 4.—COMPARISON OF EXPERIMENTAL WAVE PROFILES WITH CURVE OF SECH ²

cation of the correctness of the hyperbolic secant function in describing the profile. In this figure η/a is plotted versus x/x_c, in which x_c is the horizontal distance to the place where $\eta/a = 0.2$. The solid curve represents the equation

$$\eta/a = \operatorname{sech}^2 (1.444\, x/x_c)$$

which is merely Eq. 5 with $\frac{m}{2} = \frac{1.444}{x_c}$ so that for $x = x_c$, $\eta/a = 0.2$. The values of η/a so calculated are within about 5% of the measured values. Actually, the experimental data show a tendency towards a slightly flatter wave crest with steeper fronts, but it is felt that the precision of these preliminary measurements makes this observation inconclusive.

The characteristics of the experimental profiles in Figs. 3 and 4 are given in Table 1:

TABLE 1.—CHARACTERISTICS OF EXPERIMENTAL WAVE PROFILES

Profile number	Depth y_1 feet	Amplitude a feet	Ratio a/y_1	Distance x_c to $\eta/a = 0.2$ feet
(1)	(2)	(3)	(4)	(5)
280	0.4311	0.0992	0.230	1.505
94	0.4409	0.0804	0.180	1.653
142	0.3766	0.0604	0.160	1.487
112	0.4377	0.0503	0.115	1.925
119	0.4367	0.0351	0.080	1.940

In Fig. 3 the same data are compared with the Boussinesq and the Rayleigh theoretical profiles. Substantial agreement with the Boussinesq form is obtained. The Rayleigh profiles fall ouside the range covered by scatter and other deviations in the measured data. These curves cover the range of values

of the ratio a/y_1 below 0.23. There is evidence to indicate that at higher ratios of a/y_1, the actual profiles may be more peaked than the Boussinesq relation would indicate. The traces obtained by Mr. Russell showed this in some instances and occasional profiles published subsequently, such as those of J. M. Caldwell[15] have indicated the same characteristics. However, the exact deviation is not yet established so that the continued use of the simpler Boussinesq equation seems justified.

In the laboratory there is always some question as to whether the generated waves have attained a final permanent form as postulated in developing the theory. First, as the wave form approaches the permanent form, the deformation rate becomes very slow or the wave may possibly oscillate about the permanent shape.[12] Second, friction will cause continual attenuation even after the stable form is attained. Fortunately, in the oscillating stage, the profile cannot change markedly, and frictional damping ultimately should eliminate the oscillations. Before this occurs, the rate of change of shape is quite perceptible. Consequently, if the rate of loss in amplitude is near that expected from friction, the wave has probably reached its final form. This was the basis for considering the profiles of Figs. 3 and 4 to be representative of the true permanent form of the solitary wave.

Wave Volume.—Using the Boussinesq profile, the volume of the elevated wave per unit of crest length (ΔQ) contained between limits of $+x$ to $-x$ is

$$\Delta Q = 2\int_0^x \eta\, dx = \left[\left(\frac{16}{3} y^3{}_1\, a\right)\left(1 - \frac{\eta}{a}\right)\right]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (7)$$

and the total volume (Q) is

$$Q = \left(\frac{16}{3} y^3{}_1\, a\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

It will be noted that the fractional volume $\frac{\Delta Q}{Q}$ is a function of η/a. For the zone in which $\eta/a \geq 0.19$ the fractional volume is 0.9. This corresponds to a distance $x = \pm 2.4\, y_1$ if $a/y_1 = 0.5$ or $x = \pm 3.8\, y_1$ if $a/y_1 = 0.2$.

Particle Motion.—The solution as determined by Mr. McCowan for the horizontal and vertical components of the local liquid velocities induced by the wave's passage are given by Mr. Munk in the form

$$\frac{u}{C} = N \frac{1 + \cos\left(M\frac{z}{y_1}\right)\cosh\left(M\frac{x}{y_1}\right)}{\left\{\cos\left(M\frac{z}{y_1}\right) + \cosh\left(M\frac{x}{y_1}\right)\right\}^2} \quad \ldots\ldots\ldots\ldots (9)$$

$$\frac{w}{C} = N \frac{\sin\left(M\frac{z}{y_1}\right)\sinh\left(M\frac{x}{y_1}\right)}{\left\{\cos\left(M\frac{z}{y_1}\right) + \cosh\left(M\frac{x}{y_1}\right)\right\}^2} \quad \ldots\ldots\ldots\ldots (10)$$

[15] "Reflection of Solitary Waves," by J. M. Caldwell, *Technical Memorandum No. 11*, Beach Erosion Board, Corps of Engrs., U. S. Army, Washington, D. C., 1949.

in which u and w are the horizontal and vertical components of the velocity, and the parameters, M and N are functions of the ratio a/y_1 (Fig. 5) according to the equations

$$a/y_1 = \frac{N}{M} \tan \tfrac{1}{2} [M\ (1 + a/y_1)] \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$N = \frac{2}{3} \sin^2 \left[M \left(1 + \frac{2\,a}{3\,y_1} \right) \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

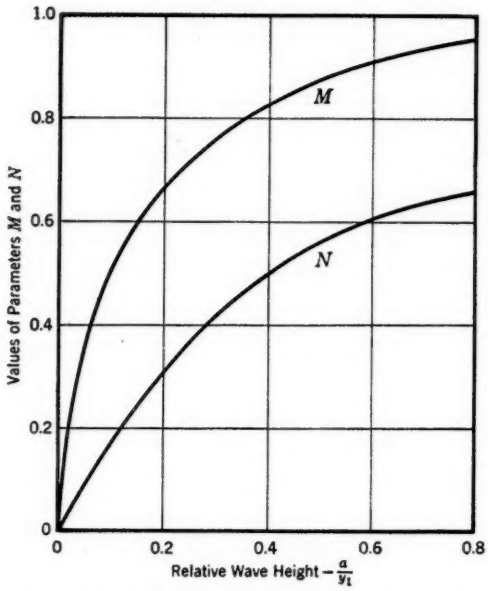

FIG. 5.—DETERMINATION OF PARAMETERS M AND N

Typical diagrams showing particle motion are given in Fig. 6 which has been adapted from a chart prepared by Mr. Munk. Study shows that these equations describe the observed characteristics of the particles that are set in motion by the oncoming wave and accelerated forward and upward until at the instant of crest passage they are at their maximum elevation and are all moving horizontally, each with its maximum velocity. The earlier solutions of Boussinesq were inaccurate to the extent of indicating finite vertical (downward) velocities beneath the crest.

After the crest passes, the particles gradually return to rest at their original elevation but displaced in the direction of motion by a certain amount, the surface liquid being displaced farther than the bottom liquid. The average value of this horizontal displacement (or mass transport) is

$$\Delta x = \frac{Q}{y_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

These particle motions and displacements have been confirmed only qualitatively. Mr. Russell approximated the orbits of particles by a curve of

versed sines, as he did the surface profile. The velocities are low, and the motions of short duration, making measurements difficult. Since these internal motions are important in the discussion of sediment transportation and of other phenomena associated with shoreward transport, return flow and drift in the surf, a more exact verification is likely to be required.

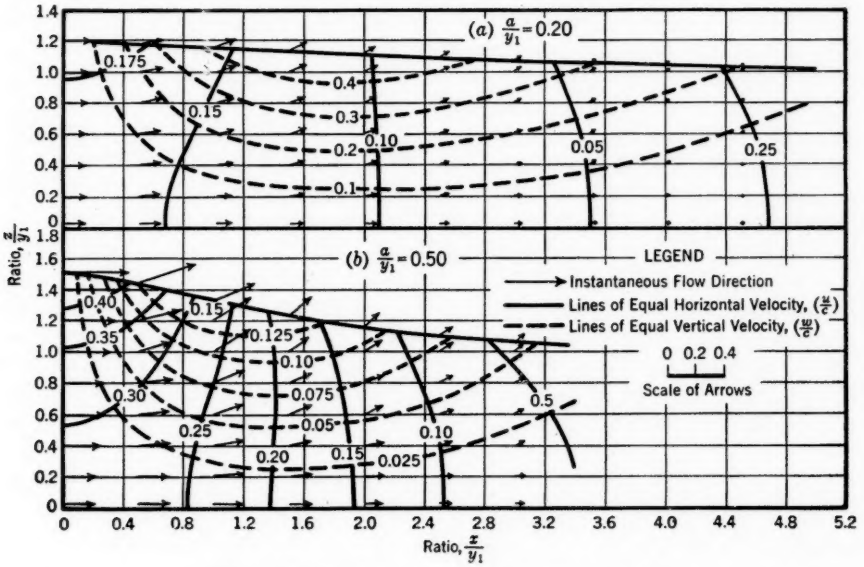

FIG. 6.—SURFACE PROFILES AND ORBITAL MOTIONS

Wave Energy.—The potential and kinetic energies of the elevated wave are approximately equal.[11] Therefore, the energy per unit length of crest (ΔE) of the liquid contained between $+ x$ and $- x$ becomes

$$\Delta E = \Delta E_p + \Delta E_k = 2 \rho g \int_0^x \frac{\eta^2}{2} dx \quad \ldots\ldots\ldots\ldots\ldots (14)$$

in which the subscripts p and k denote potential and kinetic, respectively, and ρ is the density of the liquid. Using Eqs. 5 and 6*a*,

$$E = \frac{4}{3\sqrt{3}} \rho g a^{\frac{3}{2}} y_1^{\frac{3}{2}} (2 + \eta/a) (1 - \eta/a)^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots (15)$$

and the total energy (and hence the work necessary to generate the wave)

$$E = E_p + E_k = \frac{8}{3\sqrt{3}} \rho g a^{\frac{3}{2}} y_1^{\frac{3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

As observed for the fractional volume, the fractional energy $\frac{\Delta E}{E}$ is a function of η/a (and in turn of x, y_1, and a). For $\eta/a \geq 0.19$, the fractional energy is 0.99 compared with 0.90 for the fractional volume within the same limits.

Ninety percent of the energy is contained within the limits described by $\eta/a \geq 0.46$.

The Limiting Wave.—It should be noted that a limit exists as to the amount of energy that can be propagated with a solitary wave, since in a liquid of given depth y_1, the wave will break at a certain maximum height. This limiting height can be approximated theoretically using the criterion that the liquid particle at the crest cannot travel faster than the wave itself. Thus, for $u = C$, Mr. McCowan[16] found the maximum wave to be represented by

$$(a/y_1)_{max} = \tfrac{1}{2} \tan (1 \text{ radian}) = 0.78 \ldots\ldots\ldots\ldots\ldots (17)$$

For this condition, the theoretical wave crest has a cusped apex with a 120° central angle. Messrs. Keulegan and Patterson, using the same criterion, obtained a value of 0.73 for the equation. Early experiments by Mr. Russell indicated that actual breaking should occur when the relative height is in the neighborhood of unity. All observers since have indicated a value near 0.75.

Damping Through Friction.—Since actual waves are subject to gradual damping through friction, the permanent wave is attainable only in the sense that it assumes some characteristic shape. Its amplitude continually decreases. Mr. Keulegan[13] has calculated the rate of change of amplitude through viscous energy dissipation at the solid boundaries of rectangular channels, obtaining the expression

$$\left(\frac{a}{y_1}\right)^{-\frac{1}{4}} - \left(\frac{a_0}{y_1}\right)^{-\frac{1}{4}} = \frac{1}{12}\left(1 + \frac{2\,y_1}{B}\right)\sqrt{\frac{\nu}{g^{\frac{1}{2}}\,y_1^{\frac{3}{2}}}}\,\frac{S}{y_1} \ldots\ldots\ldots (18)$$

in which ν is the kinematic viscosity of the liquid and S is the distance traveled by a wave in a rectangular channel of width B as the wave amplitude changes from a_0 to a. For a wide channel or two-dimensional case this equation reduces to

$$\left(\frac{a}{y_1}\right)^{-\frac{1}{4}} - \left(\frac{a_0}{y_1}\right)^{-\frac{1}{4}} = \frac{1}{12}\sqrt{\frac{\nu}{g^{\frac{1}{2}}\,y_1^{\frac{3}{2}}}}\,\frac{S}{y_1} \ldots\ldots\ldots\ldots (19)$$

Using values of temperature (and hence viscosity) estimated from weather records, the original attenuation data compiled by Mr. Russell checks Eq. 18. These measurements were made by allowing the waves to undergo repeated reflections from the plane ends of a 20-ft laboratory channel until the length of travel was several hundred feet. Negligible loss during reflection is assumed. This is the most extensive data available for substantiating theoretical results.

Conclusions

The extensive measurements by Mr. Russell as well as further experiments have reasonably well verified the Boussinesq-Rayleigh celerity equation. The Boussinesq profile appears to agree best with experimental results but the supporting evidence is not extensive. Efforts are underway to obtain more refined profile measurements, but major differences from the present conclusions are not expected. The expressions derived by Mr. McCowan for internal motions are

[16] "On the Highest Wave of Permanent Type," by J. McGowan, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Vol. 38, 1894, p. 351.

yet to be verified and should be given attention, particularly in view of their application to sediment transportation problems. Data on damping should also be verified and extended.

The actual need for precise verification of theoretical wave motions may be questioned since natural waves seldom, if ever, conform exactly to the pure waves discussed theoretically or investigated in the laboratory. However, experimental verification in fixing knowledge of the phenomenon will establish the reasonableness of the calculating procedures and approximations employed.

All of the results discussed are for waves in water of constant depth. In application to problems involving shoaling water, it is assumed that the characteristics at a particular still water depth are the same on the sloping as on a horizontal bottom and that transformations take place approximately in accordance with the conservation of energy. The investigation of these premises is important. Strangely enough, the work done by Mr. Russell also included a few measurements in a sloping channel. As previously noted, his observations regarding the breaking height have been modified by later observers, suggesting the desirability of checking and extending his results.

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